

Signatures of quantum fluctuations in the Dicke model by means of Rényi uncertainty

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Rényi uncertainty relations are shown to have significant importance in quantum phase transitions. They detect the quantum phase transition in the Dicke model. The Rényi entropy sum is more appropriate for describing quantum fluctuations than the standard variance products.

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I. INTRODUCTION

Uncertainty relations play a central role in quantum physics. The Heisenberg uncertainty principle is generally expressed in terms of the familiar variance-based uncertainty relation. The uncertainty principle can alternatively be formalized via an entropic uncertainty relation [1–3] that proved to be very useful in several situations [1–11]. Quantum phase transitions (QPT) result from the variation of quantum fluctuations [12]. Therefore uncertainty relations are essential in studying QPT. Variance-based uncertainty relations have been frequently applied. In this paper we argue that Rényi uncertainty relations have significant importance in quantum phase transitions. As an illustration the Dicke model is presented. The Rényi entropy sum has an abrupt change at the transition point, though it remains constant in both phases. However, the variance products are divergent in the superradiant phase.

A Rényi entropy [13] of order μ for a D -dimensional probability density function $f(r_1, \dots, r_D)$ normalized to 1 is defined by

$$R_f^\mu \equiv \frac{1}{1-\mu} \ln \int f^\mu(\mathbf{r}) d\mathbf{r}, \quad \text{for } 0 < \mu < \infty, \quad \mu \neq 1, \quad (1)$$

where \mathbf{r} stands for (r_1, \dots, r_D) . The Rényi entropy can be considered a one-parameter extension of the Shannon entropy [14] as the Rényi entropy tends to the Shannon entropy,

$$S_f = - \int f(\mathbf{r}) \ln f(\mathbf{r}) d\mathbf{r}, \quad (2)$$

when $\mu \rightarrow 1$. The Rényi entropy has been applied in several fields of quantum physics, such as quantum entanglement [15], quantum communication protocols [16], quantum correlations [17], localization properties [18], quantum revivals [19], and atomic physics [20–23].

Suppose that the probability distribution $\rho(\mathbf{r})$ can be associated with a wave function $\psi(\mathbf{r})$ as $\rho(\mathbf{r}) = |\psi(\mathbf{r})|^2$. The probability distribution $\gamma(\mathbf{p})$, on the other hand, is given by the

momentum space wave function $\phi(\mathbf{p})$, the Fourier transform of the wave function $\psi(\mathbf{r})$, as $\gamma(\mathbf{p}) = |\phi(\mathbf{p})|^2$. An uncertainty relation for the Rényi entropy sum can be found in the literature directly based on the Hausdorff-Young inequality [24–26]:

$$R_\rho^\mu + R_\gamma^\nu \geq g(\mu, \nu), \quad \frac{1}{\mu} + \frac{1}{\nu} = 2, \quad (3)$$

$$g(\mu, \nu) = \frac{D}{2} \left[\frac{1}{\mu-1} \ln \left(\frac{\mu}{\pi} \right) + \frac{1}{\nu-1} \ln \left(\frac{\nu}{\pi} \right) \right], \quad (4)$$

where D is the dimension. This uncertainty relation reaches the Shannon entropic uncertainty relation [2,27,28]

$$S_\rho + S_\gamma \geq D \ln(e\pi) \quad (5)$$

in the limit $\mu \rightarrow 1$. Equation (3) is saturated by Gaussian distribution functions; that is, the Rényi uncertainty relation is sharp.

We mention in passing that the Rényi entropy sum is the sum obtained from the phase-space marginal distributions. Further interesting inequalities can be found in Ref. [29].

In classical phase transitions there is an abrupt change in the physical properties of a system as a parameter (normally the temperature) that is responsible for the transition changes. The phenomena are due to classical fluctuations (thermal fluctuations when temperature is the parameter). Quantum phase transitions, on the other hand, occur at zero temperature. The quantum systems are at their ground states and the abrupt change in the physical properties are induced by a parameter of the system. The Hamiltonian can be written as $H = H(\lambda) = H_0 + \lambda V$, with H_0 being integrable. At the critical point $\lambda = \lambda_c$, there is an abrupt change in the symmetry of the ground-state wave function [12].

The Rényi entropy is a functional of the probability density. As the symmetry of the ground-state wave function changes, both the position and the momentum space probability densities change. Therefore there should be an abrupt change in the Rényi entropy sum too. The entropic uncertainty relation provides (see Ref. [30] and references therein) a refined version

of the Heisenberg uncertainty relation:

$$\Delta x \Delta p_x \geq \frac{1}{2} \exp[S_\rho + S_\gamma - 1 - \ln \pi] \geq \frac{1}{2}. \quad (6)$$

It gives a stronger bound for the variance product than the standard $\frac{1}{2}$. That is, the relation with the Shannon entropy sum provides a more useful form of the uncertainty relation than the one containing the variance product. We have shown that the description of the quantum phase transition in terms of the entropic uncertainty relation turns out to be more suitable than in terms of the standard variance-based uncertainty relation [31]. The importance of the entropic uncertainty measure for fluctuations has recently been emphasized [32].

In this work we will show that the uncertainty relation for the Rényi entropy sum gives a fresh insight into quantum fluctuations. To illustrate it we selected the Dicke model, which proved to be very useful in studying quantum optical [33–37], chaotic [37,38], and entanglement [39] properties. It has been realized with a superfluid gas in an optical cavity [40], and the spontaneous symmetry breaking has been observed recently [41]. There is a QPT in the $N \rightarrow \infty$ limit. It has recently been shown that there is an abrupt change in the Rényi entropy at the transition point [42] and that the transition is marked by the relative complexity measure [43].

II. RÉNYI UNCERTAINTY AND QPT IN THE DICKE MODEL

Consider an ensemble of N two-level atoms with level splitting ω_0 . The single-mode Dicke Hamiltonian has the form

$$H = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{2j}}(a^\dagger + a)(J_+ + J_-), \quad (7)$$

where J_z and J_\pm are the angular momentum operators for a pseudospin of length $j = N/2$ and a and a^\dagger are the bosonic operators of the field (the bosonic mode has a frequency ω). In the thermodynamic limit, where the number of atoms becomes infinite ($N, j \rightarrow \infty$), there is a QPT at a critical value of the atom-field coupling strength $\lambda_c = \frac{1}{2}\sqrt{\omega\omega_0}$. There are two phases: the normal phase ($\lambda < \lambda_c$) and the superradiant phase ($\lambda > \lambda_c$).

Let us consider a basis set $\{|n\rangle \otimes |j, m\rangle\}$ of the Hilbert space, with $\{|n\rangle\}_{n=0}^\infty$ being the number states of the field and $\{|j, m\rangle\}_{m=-j}^j$ being the so-called Dicke states of the atomic sector. To solve numerically the eigenproblem we have to diagonalize the matrix

$$\begin{aligned} \langle n', j', m' | H | n, j, m \rangle = & (n\omega + m\omega_0) \delta_{n',n} \delta_{m',m} + \frac{\lambda}{\sqrt{2j}} (\sqrt{n+1} \delta_{n',n+1} + \sqrt{n} \delta_{n',n-1}) (\sqrt{j(j+1) - m(m+1)} \delta_{m',m+1} \\ & + \sqrt{j(j+1) - m(m-1)} \delta_{m',m-1}), \end{aligned} \quad (8)$$

and for this purpose we truncate the bosonic Hilbert space to a finite dimension n_c large enough to have convergence in the solution (see Ref. [44] for a detailed study of the numerical problem). At this point it is important to note that time evolution preserves the parity $e^{i\pi(n+m+j)}$ of a given state $|n, j, m\rangle$. That is, the parity operator $\hat{\Pi} = e^{i\pi(a^\dagger a + J_z + j)}$ commutes with H , and both operators can then be jointly diagonalized. In particular, the ground state must be even [see Eq. (17)].

We shall make use of the Holstein-Primakoff representation [45] of the angular momentum operators J_\pm, J_z in terms of the bosonic operators, $[b, b^\dagger] = 1$, given by

$$\begin{aligned} J_+ &= b^\dagger \sqrt{2j - b^\dagger b}, \quad J_- = \sqrt{2j - b^\dagger b} b, \\ J_z &= (b^\dagger b - j). \end{aligned} \quad (9)$$

For high values of j (and fixed $b^\dagger b$), we can approximate $J_+ \simeq \sqrt{2j} b^\dagger$ and $J_- \simeq \sqrt{2j} b$, so that the atomic sector can be practically described by a harmonic oscillator, just like the field sector. Introducing then position and momentum operators for the two bosonic modes as usual,

$$\begin{aligned} X &= \frac{1}{\sqrt{2\omega}}(a^\dagger + a), \quad P_X = i\sqrt{\frac{\omega}{2}}(a^\dagger - a), \\ Y &= \frac{1}{\sqrt{2\omega_0}}(b^\dagger + b), \quad P_Y = i\sqrt{\frac{\omega_0}{2}}(b^\dagger - b), \end{aligned} \quad (10)$$

the wave function in the position representation can be represented by

$$\begin{aligned} \psi(x, y) &= \frac{\sqrt{\omega\omega_0}}{\sqrt{\pi}} e^{-\frac{1}{2}(\omega x^2 + \omega_0 y^2)} \sum_{n=0}^{n_c} \sum_{m=-j}^j c_{nm}^{(j)} \\ &\times \frac{H_n(\sqrt{\omega}x) H_{j+m}(\sqrt{\omega_0}y)}{2^{(n+m+j)/2} \sqrt{n!(j+m)!}}, \end{aligned} \quad (11)$$

where the coefficients $c_{nm}^{(j)}$ come from the numerical diagonalization of the Hamiltonian matrix (8), and we have made use of the definition of

$$\begin{aligned} \langle x | n \rangle &= \sqrt{\omega} e^{-\frac{1}{2}\omega x^2} \frac{H_n(\sqrt{\omega}x)}{\sqrt{2^n n! \sqrt{\pi}}}, \\ \langle y | j, m \rangle &= \sqrt{\omega_0} e^{-\frac{1}{2}\omega_0 y^2} \frac{H_{j+m}(\sqrt{\omega_0}y)}{\sqrt{2^{(j+m)} (j+m)! \sqrt{\pi}}} \end{aligned} \quad (12)$$

in terms of Hermite polynomials of degree n and $j+m$, respectively. This is a very convenient representation that has already been used in Ref. [37]. In the same way, the wave function in momentum representation can be written as

$$\begin{aligned} \phi(p_x, p_y) &= \frac{1}{\sqrt{\omega\omega_0\pi}} e^{-\frac{1}{2}(\frac{p_x^2}{\omega} + \frac{p_y^2}{\omega_0})} \sum_{n=0}^{n_c} \sum_{m=-j}^j (-i)^{n+m+j} c_{nm}^{(j)} \\ &\times \frac{H_n(p_x/\sqrt{\omega}) H_{j+m}(p_y/\sqrt{\omega_0})}{2^{(n+m+j)/2} \sqrt{n!(j+m)!}}, \end{aligned} \quad (13)$$

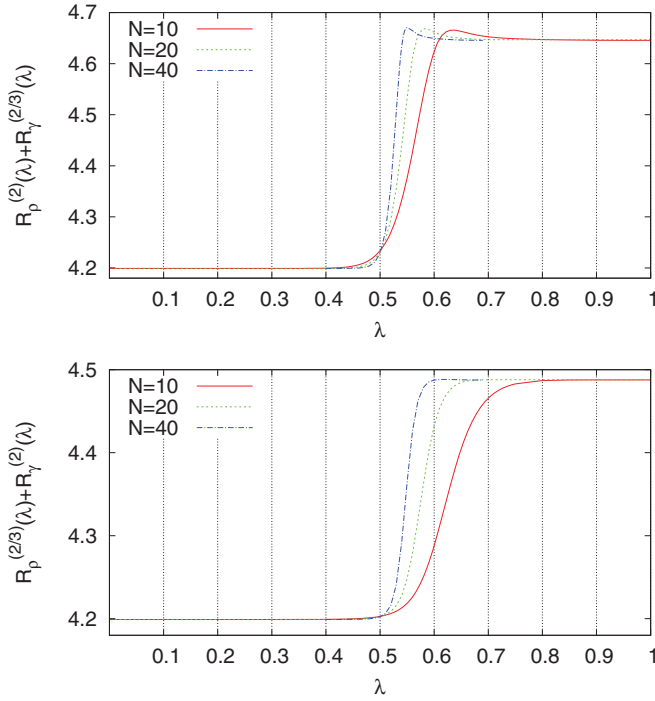


FIG. 1. (Color online) Numerical results for the Rényi uncertainty entropy sum $R_p^\mu + R_y^\nu$ for the ground state for $N = 10, 20, 40$ and $\omega_0 = \omega = 1$ ($\lambda_c = 0.5$) and for (μ, ν) equal to (top) $(2, 2/3)$ and (bottom) $(2/3, 2)$. All values are in atomic units.

where we have taken into account that the Fourier transform of $e^{-a^2 x^2/2} H_n(ax)$ is given by $(-i)^n e^{-p^2/(2a^2)} H_n(p/a)$. The position and momentum densities are given by $|\psi(x, y)|^2$ and $|\phi(p_x, p_y)|^2$ with wave functions (11) and (13). The Rényi entropies were computed numerically.

Figures 1(a) and 1(b) present the Rényi uncertainty sum $R_p^\mu + R_y^\nu$ for values of the parameter $\lambda \in [0, 1]$ and for (μ, ν) equal to $(2, 2/3)$ and $(2/3, 2)$, respectively. In the normal phase $\lambda < \lambda_c$, the uncertainty relation (3) is saturated with $R_p^2 + R_y^{2/3} \simeq 4.199 \approx g(2, 2/3) = \ln(3^3 \pi^2 / 2^2)$, thus reaching about the minimum value. In this phase, the position (momentum) space density function is a Gaussian-like one centered at the origin, as we can see in the top left (top right) panel in Fig. 2. In the superradiant phase $\lambda > \lambda_c$, the Rényi uncertainty sum reaches the constant value $R_p^2 + R_y^{2/3} \simeq 4.64 \approx \ln[54\sqrt{\pi}\Gamma(7/6)^3/\Gamma(5/3)^3]$ for values of $\lambda > \lambda_N \geq \lambda_c$, where $\lambda_N \rightarrow \lambda_c$ as $N \rightarrow \infty$ [see Eqs. (24) and (25) for general μ, ν]. This situation corresponds to a position (momentum) density function composed by two subpackets moving away from each other into different quadrants in the plane, as we can see in the bottom left panel in Fig. 2 (by a modulated Gaussian-like packet seen in the bottom right panel in Fig. 2). The transition from the first situation to the second one is around $\lambda = \lambda_c$, and it is more sudden the greater the number of particles is (see Fig. 1).

The variance products $\Delta x \Delta p_x$ and $\Delta y \Delta p_y$ were calculated in Ref. [31] for $\omega_0 = \omega = 1$, that is, $\lambda_c = 0.5$. We obtained that, for values $\lambda < \lambda_c$, the uncertainty relations reach a value of $1/2$ [that is, saturate the uncertainty relation (6)], growing as λ increases from λ_c and as N increases, such that uncertainty

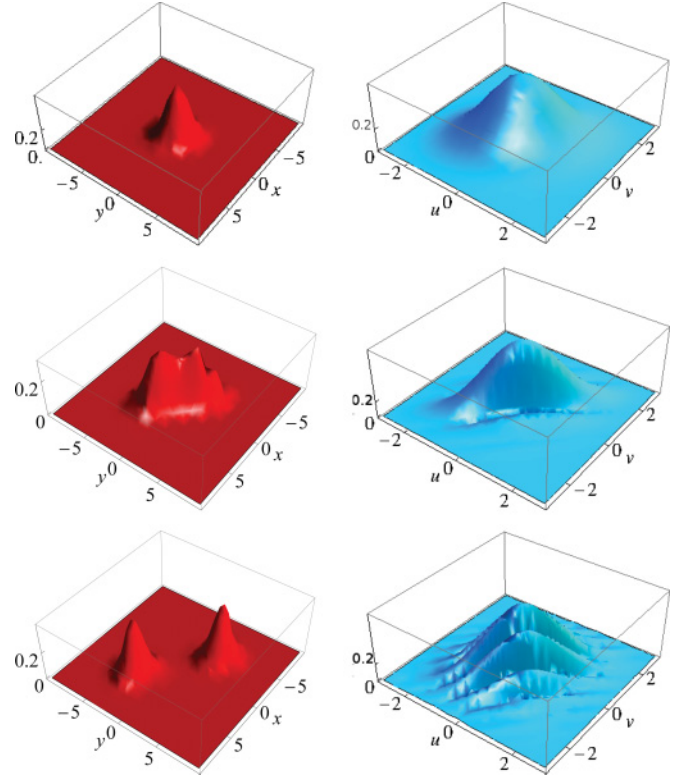


FIG. 2. (Color online) Ground-state density function in (left) position space and (right) momentum space (right) for different values of λ (from top to bottom: $\lambda = 0.3$, $\lambda = 0.55$, and $\lambda = 0.7$) for $\omega_0 = \omega = 1$ and $N = 20$. All values are in atomic units.

relations diverge for $\lambda \geq \lambda_c$ as N diverges. The most important conclusion for this analysis is that the Rényi uncertainty relations take into account the quantum fluctuations or wave-packet uncertainty (as the entropic uncertainty does [31]), whereas the variance uncertainty takes into account not only the quantum fluctuations but also the relative position of the wave-packets.

III. VARIATIONAL SCHRÖDINGER'S CAT STATES AND THE THERMODYNAMIC LIMIT

Now we present analytical expressions for uncertainty relations using trial states expressed in terms of “symmetry-adapted coherent states” introduced by Castaños *et al.* [46], which turn out to be an excellent approximation to the exact quantum solution of the ground (+) and first excited (−) states of the Dicke model. Let us denote by

$$\begin{aligned}
 |\alpha\rangle &= e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \\
 |z\rangle &= (1 + |z|^2)^{-j} e^{z J_+} |j, -j\rangle \\
 &= (1 + |z|^2)^{-j} \sum_{m=-j}^j \binom{2j}{j+m}^{1/2} z^{j+m} |j, m\rangle, \quad (14)
 \end{aligned}$$

the standard (canonical) and spin- j coherent states for the photon and the particle sectors, respectively. Using the direct

product $|\alpha, z\rangle \equiv |\alpha\rangle \otimes |z\rangle$ as a ground-state ansatz, one can easily compute the mean energy

$$\begin{aligned}\mathcal{H}(\alpha, z) &= \langle \alpha, z | H | \alpha, z \rangle \\ &= \omega |\alpha|^2 + j\omega_0 \frac{|z|^2 - 1}{|z|^2 + 1} + \lambda \sqrt{2j} (\alpha + \bar{\alpha}) \frac{\bar{z} + z}{|z|^2 + 1} \\ &= \frac{\omega}{2} (q^2 + p^2) - \omega_0 j \cos \theta + \sqrt{4j\lambda} q \sin \theta \sin \phi,\end{aligned}\quad (15)$$

which defines a four-dimensional “energy surface”; we have used quadratures $\alpha = \frac{1}{\sqrt{2}}(q + ip)$ and stereographic projection $z = \tan(\frac{\theta}{2})e^{i\phi}$ coordinates in the last equality for later convenience. Minimizing with respect to these four coordinates gives the critical points:

$$\begin{aligned}\alpha &= \alpha_0 = \begin{cases} 0, & \text{if } \lambda < \lambda_c, \\ -\sqrt{2j} \sqrt{\frac{\omega_0}{\omega}} \frac{\lambda}{\lambda_c} \sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^{-4}}, & \text{if } \lambda \geq \lambda_c, \end{cases} \\ z &= z_0 = \begin{cases} 0, & \text{if } \lambda < \lambda_c, \\ \sqrt{\frac{\frac{\lambda}{\lambda_c} - \left(\frac{\lambda}{\lambda_c}\right)^{-1}}{\frac{\lambda}{\lambda_c} + \left(\frac{\lambda}{\lambda_c}\right)^{-1}}}, & \text{if } \lambda \geq \lambda_c. \end{cases}\end{aligned}\quad (16)$$

Note that α_0 and β_0 are real, so that $p_0 = 0 = \phi_0$.

Although the direct product $|\alpha, z\rangle$ gives a good variational approximation to the ground-state mean energy in the thermodynamic limit $j \rightarrow \infty$, it does not capture the correct behavior for other ground-state properties sensitive to the parity symmetry $\hat{\Pi}$ of the Hamiltonian (7) like, for instance, uncertainty measures. This is why parity-symmetry-adapted coherent states are introduced. Indeed, a far better variational description of the ground (first excited) state is given in terms of the even (odd) parity coherent states,

$$|\alpha, z, \pm\rangle = \frac{|\alpha\rangle \otimes |z\rangle \pm |-\alpha\rangle \otimes |-z\rangle}{\mathcal{N}_{\pm}(\alpha, z)}, \quad (17)$$

obtained by applying projectors of even and odd parity $\hat{\mathcal{P}}_{\pm} = (1 \pm \hat{\Pi})$ to the direct product $|\alpha\rangle \otimes |z\rangle$. Here

$$\begin{aligned}\mathcal{N}_{\pm}(\alpha, z) &= \sqrt{2} \left[1 \pm e^{-2|\alpha|^2} \left(\frac{1 - |z|^2}{1 + |z|^2} \right)^{2j} \right]^{1/2} \\ &= \sqrt{2} [1 \pm e^{-(p^2 + q^2)} (\cos \theta)^{2j}]^{1/2}\end{aligned}\quad (18)$$

is a normalization factor. These even and odd coherent states are “Schrodinger’s cat states” in the sense that they are a quantum superposition of quasiclassical, macroscopically distinguishable states.

The new energy surface is now

$$\begin{aligned}\mathcal{H}_{\pm}(\alpha, z) &= \langle \alpha, z, \pm | H | \alpha, z, \pm \rangle \\ &= \frac{\mathcal{H}(\alpha, z) \pm \langle \alpha, z | H | -\alpha, -z \rangle}{\mathcal{N}_{\pm}(\alpha, z)^2/2},\end{aligned}\quad (19)$$

with nondiagonal elements,

$$\begin{aligned}\langle \alpha, z | H | -\alpha, -z \rangle &= e^{-2|\alpha|^2} \left(\frac{1 - |z|^2}{1 + |z|^2} \right)^{2j} \left[\omega |\alpha|^2 - j\omega_0 \frac{1 + |z|^2}{1 - |z|^2} \right. \\ &\quad \left. + \lambda \sqrt{2j} (\alpha - \bar{\alpha}) \frac{z - \bar{z}}{1 - |z|^2} \right] \\ &= e^{-(p^2 + q^2)} (\cos \theta)^{2j} \left[\frac{\omega}{2} (q^2 + p^2) - \omega j (\cos \theta)^{2j-1} \right. \\ &\quad \left. - \sqrt{4j\lambda} p \sin \theta \cos \phi \right].\end{aligned}\quad (20)$$

The more involved structure of $\mathcal{H}_{\pm}(\alpha, z)$ makes it much more difficult to obtain the new critical points $\alpha_0^{(\pm)}, z_0^{(\pm)}$, minimizing the corresponding energy surface. Instead of carrying out a numerical computation of $\alpha_0^{(\pm)}, z_0^{(\pm)}$ for different values of j and λ , we shall use the approximation $\alpha_0^{(\pm)} \approx \alpha_0, z_0^{(\pm)} \approx z_0$, which turns out to be quite good except in close vicinity to λ_c which diminishes as the number of particles $N = 2j$ increases (see Fig. 3 for the symmetric case and Ref. [47]). With this approximation, we expect a rather good agreement between our numerical and variational results except perhaps in close vicinity to λ_c .

In order to compute uncertainty relations for information entropies in position and momentum representations, we shall make use of the Holstein-Primakoff representation (9). Redefining $\beta \equiv \sqrt{2j} z$, it can be seen (see, e.g., [48,49]) that spin- j coherent states $|z\rangle$ go over to ordinary coherent states $|\beta\rangle \equiv e^{-|\beta|^2/2} e^{\beta b^\dagger} |0\rangle$ for $j \gg 1$ (when identifying

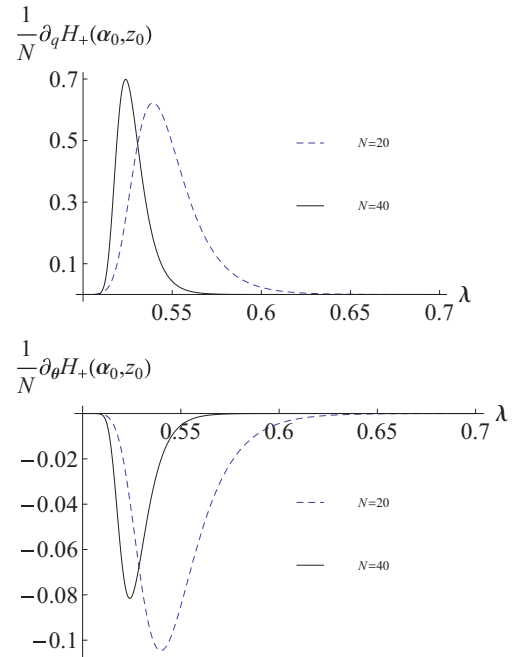


FIG. 3. (Color online) Derivative of the mean energy $\mathcal{H}_{+}(\alpha, z)$ per particle with respect to $q = \sqrt{2} \text{Re}(\alpha)$ and $\theta = 2 \arctan(|z|)$, evaluated at the critical points (16), as a function of λ for $N = 20$ (dashed line) and $N = 40$ (solid line) particles and $\omega_0 = \omega = 1$. All values are in atomic units.

$|j, -j\rangle \equiv |0\rangle$). Thus, we shall assume the approximation

$$|z\rangle \simeq |\beta\rangle, \quad (21)$$

which turns out to be a quite good estimate even for relatively small values of j . Introducing position and momentum operators for the two bosonic modes as in Eq. (10) and taking into account the position and momentum representation of an ordinary (canonical) coherent state ($\alpha = \alpha_1 + i\alpha_2$) [49],

$$\begin{aligned} \langle x|\alpha\rangle &= \left(\frac{\omega}{\pi}\right)^{1/4} e^{i\sqrt{2\omega}\alpha_2 x} e^{-(\sqrt{\omega}x - \sqrt{2}\alpha_1)^2/2}, \\ \langle p|\alpha\rangle &= \left(\frac{\omega}{\pi}\right)^{1/4} e^{i\sqrt{\frac{2}{\omega}}\alpha_1 p} e^{-(\frac{p}{\sqrt{\omega}} - \sqrt{2}\alpha_2)^2/2}, \end{aligned} \quad (22)$$

the explicit expression of the ground-state wave function $|\alpha_0, \beta_0\rangle_+$ in position $[\psi(x, y) = \langle x, y|\alpha_0, \beta_0, +\rangle]$ and momentum $[\phi(p_x, p_y) = \langle p_x, p_y|\alpha_0, \beta_0, +\rangle]$ representations can be easily obtained as

$$\begin{aligned} \psi(x, y) &= \frac{\sqrt{\omega_0\omega}}{\mathcal{N}_+(\alpha_0, \beta_0)} \left(e^{-\frac{1}{2}(\sqrt{\omega}x - \sqrt{2}\alpha_0)^2 - \frac{1}{2}(\sqrt{\omega_0}y - \sqrt{2}\beta_0)^2} \right. \\ &\quad \left. + e^{-\frac{1}{2}(\sqrt{\omega}x + \sqrt{2}\alpha_0)^2 - \frac{1}{2}(\sqrt{\omega_0}y + \sqrt{2}\beta_0)^2} \right), \\ \phi(p_x, p_y) &= \frac{2/\sqrt{\omega_0\omega}}{\mathcal{N}_+(\alpha_0, \beta_0)} e^{-\frac{p_x^2}{2\omega} - \frac{p_y^2}{2\omega_0}} \\ &\quad \times \cos \left[\sqrt{2} \left(\frac{p_x}{\sqrt{\omega}}\alpha_0 + \frac{p_y}{\sqrt{\omega_0}}\beta_0 \right) \right], \end{aligned} \quad (23)$$

where now $\mathcal{N}_+(\alpha_0, \beta_0) = [2\pi(1 + e^{-2\alpha_0^2 - 2\beta_0^2})]^{1/2}$ is a new normalization factor. Note that ψ and ϕ depend on j and λ through α_0 and β_0 . Moreover, note also that for $\lambda > \lambda_c$ the ground-state density function $\rho(x, y) = |\psi(x, y)|^2$ splits into two Gaussian packets centered at antipodal points $\sqrt{2}(\alpha_0, \beta_0)$ and $-\sqrt{2}(\alpha_0, \beta_0)$ in the x - y plane. The packets move away from each other for increasing j above the critical point $\lambda > \lambda_c$. In momentum space, $\gamma(p_x, p_y) = |\phi(p_x, p_y)|^2$ is a Gaussian modulated by a cosine function which oscillates rapidly for high j for $\lambda > \lambda_c$. This behavior is also captured by the numerical solution, as depicted in Fig. 2.

This particular ground-state wave function structure leads to a Heaviside (step) function behavior of the Rényi entropy in the position,

$$R_\rho^\mu = \begin{cases} \ln(\mu^{\frac{1}{\mu-1}}\pi), & \text{if } \lambda < \lambda_c, \\ \ln(2\mu^{\frac{1}{\mu-1}}\pi), & \text{if } \lambda \geq \lambda_c, \end{cases} \quad (24)$$

and momentum,

$$R_\gamma^\nu = \begin{cases} \ln(\nu^{\frac{1}{\nu-1}}\pi), & \text{if } \lambda < \lambda_c, \\ \ln\left[\left(\frac{\Gamma(2\nu+1)}{\Gamma^2(\nu+1)2^\nu}\right)^{\frac{1}{1-\nu}}\pi\right], & \text{if } \lambda \geq \lambda_c, \end{cases} \quad (25)$$

representations in the thermodynamic limit ($j \rightarrow \infty$). This behavior can be inferred from Fig. 4. In the normal phase, inequality (3) saturates [that is, the total entropy is exactly $g(\mu, \nu)$] because the ground-state wave function (23) is a Gaussian centered at the origin in the position and momentum representations. Above the critical point λ_c , the original Gaussian wave packet splits into two subpackets with negligible overlap, which results in a sudden rise of the total Rényi

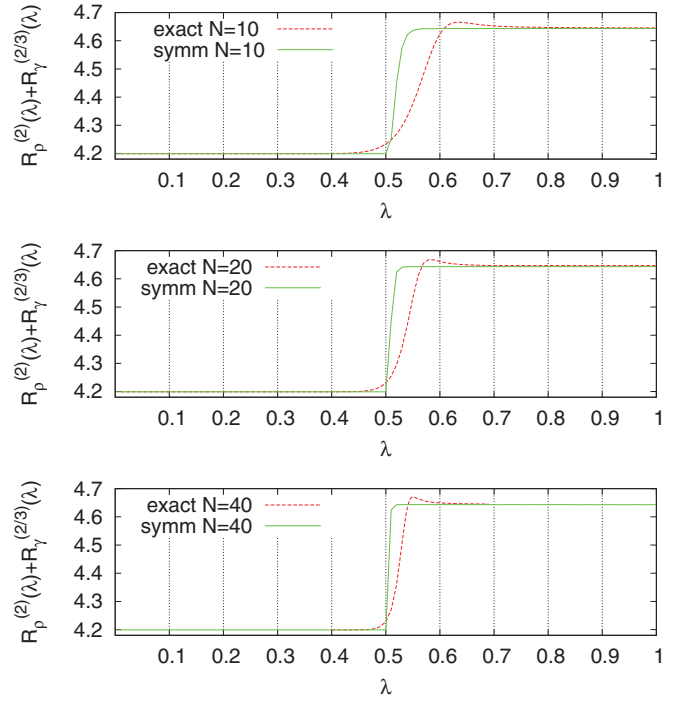


FIG. 4. (Color online) Analytical approximation and numerical (exact) results for the ground-state Rényi uncertainty entropy sum $R_\rho^\mu + R_\gamma^\nu$ for (top) $N = 10$, (middle) 20, and (bottom) 40 and $\omega_0 = \omega = 1$. All values are in atomic units.

entropy. In the limit $\mu, \nu \rightarrow 1$, we recover the expression for the Shannon entropy given in Ref. [31]:

$$S_\rho + S_\gamma = \begin{cases} S^{\text{normal}} = \ln(e\pi)^2 \simeq 4.29, & \text{if } \lambda < \lambda_c, \\ S^{\text{super}} = \ln[(2\pi)^2 e] \simeq 4.68, & \text{if } \lambda \geq \lambda_c. \end{cases} \quad (26)$$

For general (μ, ν) , we can calculate the uncertainty gap

$$\Delta U(\mu, \nu) \equiv (R_\rho^\mu + R_\gamma^\nu)_{\text{super}} - (R_\rho^\mu + R_\gamma^\nu)_{\text{normal}}$$

in the thermodynamic limit, giving

$$\Delta U(\mu, \nu) = \frac{1}{1-\nu} \ln \left(\frac{2\Gamma(\nu+1/2)}{\Gamma(\nu+1)\sqrt{\pi}} \right) \quad \text{with } \nu \in [1/2, \infty). \quad (27)$$

This gap is a decreasing function of ν , so the biggest gap $\Delta U(\mu, \nu) = \ln(\frac{2^4}{\pi^2})$ will be reached when $(\mu, \nu) = (\infty, 1/2)$. Thus, this combination of coefficients (μ, ν) , which gives the uncertainty measure $R_\rho^\infty + R_\gamma^{1/2}$, with

$$R_\rho^\infty = \lim_{\mu \rightarrow \infty} R_\rho^\mu = -\ln[\max \rho(\mathbf{r})], \quad (28)$$

is the more suitable one to detect a phase transition in this model. It is an important point that the better (μ, ν) combination depends on the model.

We would also like to point out that the Heaviside (step) function behavior of $R_\rho^\mu + R_\gamma^\nu$ should also appear in other quantum systems where a single wave packet splits into several subpackets above a critical value λ_c of some parameter λ of the theory. In particular, for M identical subpackets with negligible overlap, one can see that the Rényi entropy in position representation R_ρ^μ increases by an amount of $\ln(M)$.

For completeness, we also give the explicit expressions for expectation values,

$$\begin{aligned}\langle a \rangle_+ &= \langle b \rangle_+ = \langle a^\dagger \rangle_+ = \langle b^\dagger \rangle_+ = 0, \\ \langle a^\dagger a \rangle_+ &= -\alpha_0^2 [1 - 4\pi/\mathcal{N}_+(\alpha_0, \beta_0)^2], \\ \langle b^\dagger b \rangle_+ &= -\beta_0^2 [1 - 4\pi/\mathcal{N}_+(\alpha_0, \beta_0)^2],\end{aligned}\quad (29)$$

and fluctuations,

$$\begin{aligned}\Delta x &= \sqrt{\frac{\frac{1}{2} + \frac{4\pi\alpha_0^2}{\mathcal{N}_+(\alpha_0, \beta_0)^2}}{\omega}}, \\ \frac{\Delta p_x}{\sqrt{\omega}} &= \sqrt{2\alpha_0^2 \left(\frac{2\pi}{\mathcal{N}_+(\alpha_0, \beta_0)^2} - 1 \right) + \frac{1}{2}}, \\ \Delta y &= \sqrt{\frac{\frac{1}{2} + \frac{4\pi\beta_0^2}{\mathcal{N}_+(\alpha_0, \beta_0)^2}}{\omega_0}}, \\ \frac{\Delta p_y}{\sqrt{\omega_0}} &= \sqrt{2\beta_0^2 \left(\frac{2\pi}{\mathcal{N}_+(\alpha_0, \beta_0)^2} - 1 \right) + \frac{1}{2}}.\end{aligned}\quad (30)$$

Other interesting physical quantities are also the atomic inversion,

$$\langle J_3 \rangle_+/j = \langle b^\dagger b \rangle_+/j - 1 \xrightarrow{j \rightarrow \infty} 2 \frac{\frac{\lambda}{\lambda_c} - \left(\frac{\lambda}{\lambda_c}\right)^{-1}}{\frac{\lambda}{\lambda_c} + \left(\frac{\lambda}{\lambda_c}\right)^{-1}} - 1,$$

and the mean photon number,

$$\langle a^\dagger a \rangle_+/j \xrightarrow{j \rightarrow \infty} 2 \frac{\omega_0}{\omega} \left(\frac{\lambda}{\lambda_c} \right)^2 \left[1 - \left(\frac{\lambda}{\lambda_c} \right)^{-4} \right],$$

for $\lambda > \lambda_c$ in the thermodynamic limit.

IV. CONCLUSIONS

Summarizing, the Rényi entropy sum can be considered a measure of fluctuations. In this paper we demonstrate that it remains constant even in the superradiant phase. The variance products are divergent in the superradiant phase; therefore the uncertainty relation for the Rényi entropy sum provides a description with better quality.

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